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APPLICATIONS OF VARIATIONAL INEQUALITIES TO THE EXISTENCE THEOREM ON QUADRATURE DOMAINS

BY

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ABSTRACT. In this paper we shall study quadrature domains for the class of subharmonic functions. By using the theory of variational inequalities, we shall give a new proof of the existence and uniqueness theorem. As an application, we deal with Hele-Shaw flows with a free boundary and show that their two weak solutions, one of which was defined by the author using quadrature domains and the other was defined by Gustafsson [3] using variational inequalities, are identical with each other.

Introduction. In a previous paper [7], the author has defined the quadrature domains of positive measures for the class of subharmonic functions and studied their applications to complex function theory.

Let ν be a finite positive measure on the two-dimensional Euclidean space $\mathbf{R}^2$. Let $SL^1(\Omega)$ be the class of subharmonic functions in an open set Ω which are integrable with respect to the two-dimensional Lebesgue measure m. A nonempty open set Ω is called a quadrature domain of ν for class SL^1 if

(Qi) ν is concentrated in Ω, namely, $\nu(\Omega^c) = 0$, where Ω^c denotes the complement of Ω,

(Qii) $\int_\Omega s^+ \, d\nu < \infty$ and $\int_\Omega s \, d\nu \leqslant \int_\Omega s \, dm$ for every $s \in SL^1(\Omega)$, where $s^+ = \max\{s, 0\}$.

(Qiii) $m(\Omega) < \infty$.

Let us denote by $Q(\nu, SL^1)$ the class of all quadrature domains of ν for class SL^1. The class $Q(\nu, SL^1)$ may be empty. Let W be an open set with finite area and let f be a nonnegative bounded integrable function in $\mathbf{R}^2$ satisfying $f = 0$ a.e. in W^c. If $\sup_W f < 1$, then $Q(fm, SL^1) = \varnothing$. The class $Q(\chi_W m, SL^1)$ consists of all open sets Ω satisfying $\chi_W = \chi_\Omega$ a.e. in $\mathbf{R}^2$, where χ_W denotes the characteristic function of W, namely, $\chi_W(x) = 1$ for $x \in W$ and $\chi_W(x) = 0$ for $x \notin W$.

On the contrary, the author has already proved the following theorem (cf. [7, Theorem 3.7]):

THEOREM 1. *Let f be a bounded integrable function in $\mathbf{R}^2$ such that $f \geqslant 1$ a.e. in a connected open set W with finite area, $f = 0$ a.e. in W^c and $\int f \, dm > m(W)$, then $Q(fm, SL^1) \neq \varnothing$ and there exists a minimum domain $\tilde{W}$ in $Q(fm, SL^1)$, namely, $\Omega \in Q(fm, SL^1)$ if and only if $\tilde{W} \subset \Omega$ and $m(\Omega \setminus \tilde{W}) = 0$.*

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The main purpose of this paper is to give this theorem a new proof by using variational inequalities.

Recently, Gustafsson [3] has used variational inequalities to solve a moving boundary problem for Hele-Shaw flows. As a corollary, he has proved the existence of quadrature domains of a finite sum of positive point masses for the class of all complex-valued analytic integrable functions [3, Corollary 16.1].

To obtain the result, Gustafsson has used the fact that the boundaries of the above quadrature domains are algebraic curves, so this is a very special case in the theory of quadrature domains. In this paper, we shall deal with a general case stated as in the theorem.

1. Variational inequalities. In this section, we shall show our theorem for a special function f by using variationial inequalities. We assume that W is a bounded open set $\mathbf{R}^2$ and f is a bounded integrable function with $f > 1$ a.e. in W and $f = 0$ a.e. in W^c. The proof will be divided into four steps. Each step is given as a proposition.

For a real-valued bounded integrable function g in $\mathbf{R}^2$ with compact support, we define the logarithmic potential U^g of g by

$$U^g(y) = \int(-\log|x-y|)g(x)\,dm(x),$$

where $|x-y| = (\sum_{j=1}^2(x_j - y_j)^2)^{1/2}$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It is known that U^g is of class C^1 in $\mathbf{R}^2$ and $\Delta U^g = -2\pi g$ in the sense of distributions. First we shall show the following lemma:

LEMMA 1. *Let $\Omega \in Q(fm, SL^1)$. Then Ω is bounded.*

PROOF. Let f_1 be a nonnegative integrable function in $\mathbf{R}^2$ such that $f_1 \geqslant 1$ a.e. in an open set W_1 and $f_1 = 0$ a.e. in W_1^c. Let Ω_1 satisfy $m(\Omega_1) < \infty$ and

$$(1) \qquad \int_{W_1} sf_1\,dm \leqslant \int_{\Omega_1} s\,dm$$

for every $s \in SL^\infty(W_1 \cup \Omega_1)$, where $SL^\infty(W_1 \cup \Omega_1)$ denotes the class of all bounded subharmonic functions in $W_1 \cup \Omega_1$.

First we show that if Ω_1 is a bounded open set with smooth boundary, then W_1 is contained in the bounded open set G whose boundary is the outer boundary of Ω_1. Assume $W_1 \setminus G \neq \varnothing$. Then $(\partial W_1) \setminus \overline{G} \neq \varnothing$.

Choose a point $x_0 \in (\partial W_1) \setminus \overline{G}$ and $r > 0$ so that $\mathrm{Cap}(W_1^c \cap \overline{B(x_0; r)}) > 0$ and $\overline{G} \cap \overline{B(x_0; r)} = \varnothing$, where $B(x_0; r) = \{x \in \mathbf{R}^2 \,|\, |x - x_0| < r\}$. Let μ be the equilibrium distribution of $E = W_1^c \cap \overline{B(x_0; r)}$ and let u be the conductor potential of E, namely,

$$u(y) = \int_E (-\log|x-y|)\,d\mu(x).$$

Then u is bounded from above and harmonic in E^c. Set $\alpha = \sup_{\overline{G}} u$ and $s = \max\{u, \alpha\} - \alpha$. Then $s \in SL^\infty(W_1 \cup \Omega_1)$, $\int_{W_1} sf_1\,dm > 0$ and $\int_{\Omega_1} s\,dm = 0$. This contradicts (1) and hence $W_1 \subset G$.

Since W is bounded, we can choose a ball B centered at the origin and $M > 1$ so that $f \leqslant M\chi_B$ a.e. in $\mathbf{R}^2$. Set $f_1 = \chi_\Omega + M\chi_B - f$ and $W_1 = \Omega \cup B$. Let Ω_1 be a ball centered at the origin such that $m(\Omega_1) = Mm(B)$. We shall show that Ω_1 satisfies (1). Then, by the above argument, we see that $W_1 = \Omega \cup B$ is contained in $G = \Omega_1$, namely, Ω is bounded.

To show that Ω_1 satisfies (1), let $s \in SL^\infty(W_1 \cup \Omega_1)$. Let s^* be a function in $SL^\infty(W_1 \cup \Omega_1)$ which is harmonic in Ω, and satisfies $s \leqslant s^*$ in Ω and $s = s^*$ a.e. in $\Omega_1 \setminus \Omega$; note here that $W_1 \cup \Omega_1 = \Omega \cup \Omega_1$. Then

$$\int s^*(\chi_\Omega + M\chi_B - f)\, dm = \int s^* M\chi_B\, dm \leqslant \int_{\Omega_1} s^*\, dm.$$

Subtracting $\int (s^* - s)\chi_{\Omega\cap\Omega_1}\, dm$ from both sides, we obtain

$$\int_{W_1} s f_1\, dm \leqslant \int \left\{ s^*\chi_{\Omega\setminus\Omega_1} + s\chi_{\Omega\cap\Omega_1} + s^*(M\chi_B - f) \right\} dm \leqslant \int_{\Omega_1} s\, dm.$$

This completes the proof.

Proposition 1. *Let $\Omega \in Q(fm, SL^1)$ and set $u = -1/(2\pi)U^{\chi_\Omega - f}$. Then u and Ω satisfy*

(i) $u \geqslant 0$ *in* $\mathbf{R}^2$,

(ii) $u = 0$ *in* Ω^c,

(iii) $\Delta u = \chi_\Omega - f$ *in the sense of distributions.*

Proof. Since W and Ω are both bounded, $\chi_\Omega - f$ has a compact support. Hence u is well defined and (iii) is evident.

For every $y \in \mathbf{R}^2$, $\log|x - y| \in SL^1(\Omega)$ and so

$$U^{\chi_\Omega - f}(y) = \int_W (\log|x - y|) f\, dm(x) - \int_\Omega \log|x - y|\, dm(x) \leqslant 0.$$

Hence $u \geqslant 0$ in $\mathbf{R}^2$. If $y \notin \Omega$, then both $\log|x - y|$ and $-\log|x - y|$ belong to $SL^1(\Omega)$. Hence $u(y) = -1/(2\pi)U^{\chi_\Omega - f}(y) = 0$.

Let B be a large open ball centered at the origin such that $\overline{W} \subset B$, and let $g_B(x, y)$ be the Green function in B of the Laplacian relative to the first boundary condition with pole at y.

Set

$$\psi(y) = -\frac{1}{2\pi}\int_B g_B(x, y)(f - \chi_B)(x)\, dm(x).$$

Then $\psi \in C^1(B)$ and ψ can be extended onto a neighborhood of $\overline{B}$ so that the extension, we also write it by ψ, is of class C^1 in the neighborhood. It is easy to show that $\psi = 0$ on ∂B and $\Delta\psi = f - \chi_B$ in B in the sense of distributions.

Let us denote by $H^1(B)$ the Sobolev space $H^{1,2}(B)$ with the norm

$$\|u\|_{H^{1,2}(B)} = \sum_{0 \leqslant |\alpha| \leqslant 1} \|D^\alpha u\|_{L^2(B)}$$

and denote by $H_0^1(B)$ the closure of $C_0^\infty(B)$ in the above norm. According to Poincaré's inequality, it is well known that $\|\nabla u\|_{L^2(B)}$ is a norm equivalent to the above norm for $H_0^1(B)$. In what follows, we shall understand that $H_0^1(B)$ is the Hilbert space with the norm $\|u\| = \|\nabla u\|_{L^2(B)}$ (see, e.g. Kinderlehrer and Stampacchia [5, Chapter II, §4]). We note here that $\psi \in H_0^1(B)$.

Let us consider the following variational problem: Minimize ||h|| in the closed convex set $K = \{h \in H_0^1(B) \mid h \geqslant \psi$ a.e. in $B\}$. The extremal function $v(\psi)$ exists and is determined uniquely. It is easy to show that $v = v(\psi)$ can be characterized by

(Vi) $v \in K$,

(Vii) $\int_B \nabla(h - v)\nabla v \, dm \geqslant 0$ for every $h \in K$.

PROPOSITION 2. *If $u \in H_0^1(B)$ and an open subset Ω of B satisfy*

(i)′ $u \geqslant 0$ *a.e. in B,*

(ii)′ $u = 0$ *a.e. in $B \setminus \Omega$,*

(iii)′ $\Delta u = \chi_\Omega - f$ *in B in the sense of distributions,*

then $v = u + \psi$ satisfies (Vi) *and* (Vii).

PROOF. It is evident that (Vi) follows from (i)′. Since $\Delta v = \Delta u + \Delta\psi = \chi_\Omega - \chi_B \in L^2(B)$, we have

$$\int_B \nabla(h - v)\nabla v \, dm = -\int_B (h - v)\Delta v \, dm = \int_{B\setminus\Omega} (h - v) \, dm$$

for every $h \in H_0^1(B)$. The condition (Vii) follows from the following equalities:

$$\int_{B\setminus\Omega} (h - v) \, dm = \int_{B\setminus\Omega} \{(h - \psi) - u\} \, dm = \int_{B\setminus\Omega} (h - \psi) \, dm.$$

PROPOSITION 3. *If $v \in H_0^1(B)$ satisfies* (Vi) *and* (Vii), *then $u = v - \psi \in C^1(\overline{B})$ and $u = 0$ on ∂B. The function u and $\Omega = \{x \in B \mid u(x) > 0\}$ satisfy* (i)′ *to* (iii)′ *in Proposition* 2.

PROOF. The condition (i)′ follows from (Vi).

Since $\psi \in H_0^1(B)$ and $\Delta\psi = f - \chi_B \in L^\infty(B)$, $\psi \in H^{2,s}(B)$ for every s with $1 < s < \infty$ (see, e.g. Kinderlehrer and Stampacchia [5, Chapter II, Theorem 4.10]). Hence $v \in H^{2,s}(B) \cap C^{1,\lambda}(\overline{B})$ for every s with $2 < s < \infty$, where $\lambda = 1 - 2/s$ (cf. e.g. [5, Chapter IV, Theorem 2.3]). Hence $u = v - \psi \in C^1(\overline{B})$ and $u = 0$ on ∂B. Set $\Omega = \{x \in B \mid u(x) > 0\}$. Then (ii)′ is satisfied evidently.

Let ρ be a function of class C_0^∞ with $0 \leqslant \rho \leqslant 1$ in B. Since $v \pm \rho u \in K$ and $\Delta v \in L^2(B)$, by (Vii), we have

$$\int_B \rho u \Delta v \, dm = \int_B \nabla(-\rho u)\nabla v \, dm = 0$$

for every ρ. Hence $u\Delta v = 0$ a.e. in B and so $\Delta u + \Delta\psi = \Delta v = 0$ a.e. in Ω. This implies that $\Delta u = 1 - f$ a.e. in Ω.

On $I = B \setminus \Omega$, by definition, $u = 0$ and so $\Delta u = 0$ a.e. (see, e.g. [5, Chapter II, Appendix A, Lemma A4]). By (Vii), we have

$$-\int \rho \Delta v \, dm \geqslant 0$$

for every $\rho \in H_0^1(B)$ with $\rho \geqslant 0$. Hence $\Delta v \leqslant 0$ a.e. in B and so $f - \chi_B = \Delta\psi = \Delta v \leqslant 0$ a.e. on I. This implies that $m(W \setminus \Omega) = 0$ since $f > 1$ a.e. in W. Hence $\Delta u = 0 = -f$ a.e. on I. Combining this with $\Delta u = 1 - f$ a.e. in Ω, we obtain (iii)′.

LEMMA 2. *Let Ω be an open set stated as in Proposition* 3. *Then we can choose a large open ball B so that $\overline{\Omega} \subset B$.*

PROOF. Take a ball B_0 and $M > 1$ so that $f \leqslant M\chi_{B_0}$. Then it is easily verified that $Q(M\chi_{B_0}m, SL^1)$ consists of the ball B_1 which satisfies $m(B_1) = Mm(B_0)$ and has the same center as B_0 (see [7, §1]). Choose a ball B so that $\overline{B}_1 \subset B$ and fix it.

As before Proposition 2, let us consider the obstacle problem and write $\psi = \psi(f)$, $K = K(f)$ and $v = v(f)$. For the corresponding function and the open set stated as in Proposition 3, we write $u = u(f)$ and $\Omega = \Omega(f)$, respectively. Then, by Propositions 1 and 2, $\Omega(M\chi_{B_0}) = B_1$. Hence it is sufficient to show that if $f \leqslant f_1$, then $u(f) \leqslant u(f_1)$.

First we show that if $h \in K(f)$ and $\Delta h \leqslant 0$ a.e. in B, then $v(f) \leqslant h$ a.e. in B. Set $w = h - v(f)$. Then, as we have seen in the proof of Proposition 3, $\Delta v(f) = 0$ a.e. in Ω. Hence $\Delta w = \Delta h \leqslant 0$ a.e. in Ω and so w is superharmonic in Ω. Since $w = h - \psi(f) \geqslant 0$ a.e. in $B \setminus \Omega$ and $w \in H_0^1(B)$, we have $w \geqslant 0$ a.e. in B, namely, $v(f) \leqslant h$ a.e. in B.

Now we shall show that if $f \leqslant f_1$, then $u(f) \leqslant u(f_1)$. Let $h = u(f_1) + \psi(f)$. Then $h \in K(f)$ and $\Delta h = \Delta u(f_1) + \Delta\psi(f) \leqslant \Delta u(f_1) + \Delta\psi(f_1) = \Delta v(f_1) \leqslant 0$ a.e. in B. Hence, by the above argument, we see that $u(f) + \psi(f) = v(f) \leqslant h = u(f_1) + \psi(f)$. Therefore $u(f) \leqslant u(f_1)$. This completes the proof.

PROPOSITION 4. *If $u \in H_0^1(B)$ and an open set Ω with $\overline{\Omega} \subset B$ satisfy* (i)′ *to* (iii)′ *in Proposition* 2, *then $u \in C^1(\overline{B})$, and $\tilde{W} = \{x \in B \mid u(x) > 0\}$ is the minimum open set in $Q(fm, SL^1)$.*

PROOF. The function $u(x) + 1/(2\pi)\int_B g_B(y, x)(\chi_\Omega - f)(y)\, dm(y)$ belongs to $H_0^1(B)$ and is harmonic to B. This implies that it is identically equal to zero and so $u(x) = -1/(2\pi)\int_B g_B(y, x)(\chi_\Omega - f)(y)\, dm(y)$. Since $\overline{W} \cup \overline{\Omega} \subset B$, by (iii)′,

$$\int_B \left\{ g_B(y, x) - \log\frac{1}{|y - x|} \right\} (\chi_\Omega - f)(y)\, dm(y)$$
$$= -\int_B \nabla \left\{ g_B(y, x) - \log\frac{1}{|y - x|} \right\} \nabla u(y)\, dm(y).$$

The above is equal to

$$\int_B \Delta \left\{ g_B(y, x) - \log\frac{1}{|y - x|} \right\} u(y)\, dm(y),$$

because $u \in H_0^1(B)$. Since $g_B(y, x) - \log(1/|y - x|)$ is harmonic, the above integral is equal to zero. Hence $u = -1/(2\pi)U^{\chi_\Omega - f}$, $u \in C^1(\overline{B})$ and $u \geqslant 0$ in B.

Set $\tilde{W} = \{x \in B \mid u(x) > 0\}$. Then, by (i)′ and (ii)′, we have $\chi_{\tilde{W}} \leqslant \chi_\Omega$ a.e. in B. Since $\Delta u = 0$ a.e. in $B \setminus \tilde{W}$ (see, e.g. Kinderlehrer and Stampacchia [**5**, Chapter II, Appendix A, Lemma A4]) and $f > 1$ a.e. in W, by (iii)′, we see that $\chi_{W \cup \Omega} \leqslant \chi_{\tilde{W}}$ a.e. in B. Hence $\chi_{\tilde{W}} = X_\Omega$ a.e. in B.

Next let us show $\tilde{W} \in Q(fm, SL^1)$. In what follows, for the sake of simplicity, set $g = \chi_{\tilde{W}} - f$. Let $y \in B \setminus \tilde{W}$. Then $u(y) = 0$. Since u is of class C^1 and u attains its minimum at y, $\partial u/\partial x_j(y) = 0$, $j = 1, 2$. Hence $U^g = -2\pi u = 0$ and $\partial U^g/\partial x_j = -2\pi\partial u/\partial x_j = 0$ in $B \setminus \tilde{W}$.

Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence of C^∞-functions in $\tilde{W}$ such that $0 \leqslant \omega_n \leqslant 1$, $\omega_n = 0$ in a neighborhood of $\partial\tilde{W}$, $\omega_n = 1$ outside a neighborhood of $\partial\tilde{W}$, $\lim_{n\to\infty}\omega_n(x) = 1$ for all $x = (x_1, x_2) \in \tilde{W}$, and

$$|D^\alpha\omega_n(x)| \leqslant A_\alpha n^{-1}\delta(x)^{-|\alpha|}\left(\log\frac{1}{\delta(x)}\right)^{-1}$$

for all $x \in \tilde{W}$ and all multi-indices α, where A_α denotes a constant depending only on α, and $\delta(x)$ denotes the minimum of e^{-2} and the distance from x to $\partial\tilde{W}$. For the existence of the above sequence $\{\omega_n\}$, see Hedberg [4, p. 13, Lemma 4].

It follows that

$$\frac{\partial^2}{\partial x_j^2}(U^g\omega_n) = \frac{\partial^2 U^g}{\partial x_j^2}\omega_n + 2\frac{\partial U^g}{\partial x_j}\frac{\partial\omega_n}{\partial x_j} + U^g\frac{\partial^2\omega_n}{\partial x_j^2},$$

$$\Delta U^g = \sum_j \frac{\partial^2}{\partial x_j^2}U^g = -2\pi g$$

in the sense of distributions. Since

$$\frac{\partial U^g}{\partial x_j}(x) - \frac{\partial U^g}{\partial x_j}(y) = O\left(|x-y|\log\frac{1}{|x-y|}\right), \qquad j = 1,2,$$

for every pair of points x and y with $|x-y| < e^{-2}$,

$$U^g(x) = O\left(\delta^2(x)\log\frac{1}{\delta(x)}\right),$$

$$\frac{\partial U^g}{\partial x_j}(x) = O\left(\delta(x)\log\frac{1}{\delta(x)}\right), \qquad j = 1,2,$$

in a neighborhood of each boundary point of $\tilde{W}$. Hence

$$\int_{\tilde{W}} sg\,dm = \lim_{n\to\infty}\int_{\tilde{W}} sg\omega_n\,dm = -\frac{1}{2\pi}\lim_{n\to\infty}\int_{\tilde{W}} s\Delta(U^g\omega_n)\,dm \tag{2}$$

for every $s \in L^1(\tilde{W})$. If s is subharmonic in $\tilde{W}$, then $\Delta s \geqslant 0$ in the sense of distributions. Let φ be a nonnegative C_0^∞-function of $|x|$ in $\mathbf{R}^2$ such that $\int\varphi\,dm = 1$ and set $s_\varepsilon(x) = \int s(x-\varepsilon y)\varphi(y)\,dm(y)$ for $\varepsilon > 0$. Then s_ε is a subharmonic C^∞-function on a given compact subset of $\tilde{W}$ for every sufficiently small $\varepsilon > 0$, and $s_\varepsilon \downarrow s$ as $\varepsilon \downarrow 0$ on the compact set. Since $U^g = -2\pi u \leqslant 0$, by letting ε tend to 0, we see that

$$\int_{\tilde{W}} sg\,dm \geqslant 0$$

for every $s \in SL^1(\tilde{W})$. Hence $\tilde{W} \in Q(fm, SL^1)$. Let $\Omega \in Q(fm, SL^1)$. Then, by Proposition 1 and the above argument, we see that $\chi_\Omega = \chi_{\tilde{W}}$ a.e. in $\mathbf{R}^2$. If $y \notin \Omega$, then $-\log|x-y| \in SL^1(\Omega)$ and so

$$0 \leqslant \int(-\log|x-y|)(\chi_\Omega - f)(x)\,dm(x) = -2\pi u(y).$$

Hence $u(y) = 0$, namely, $y \notin \tilde{W}$. Therefore $\tilde{W} \subset \Omega$ for every $\Omega \in Q(fm, SL^1)$. The proof is now complete.

Thus we have proved our theorem for the function f given at the beginning of this section. From (2), we have an additional result which is also true for the function f as in Theorem 2.

COROLLARY. *Let* $\Omega \in Q(fm, SL^1)$ *and* $s \in SL^1(\Omega)$. *Then*

$$\int_W sf\,dm = \int_\Omega s\,dm$$

if and only if s *is harmonic in* $\tilde{W}$.

2. Proof of the theorem. In this section, we assume that W is an open set in $\mathbf{R}^2$ with finite area and f is a bounded integrable function with $f \geq 1$ a.e. in W, $f = 0$ a.e. in W^c and $\int_O f\,dm > m(O)$ for every connected component O of W. We shall show the following as our main theorem:

THEOREM 2. *Let* f *and* W *be as above. Then* $Q(fm, SL^1) \neq \varnothing$ *and there exists a minimum domain* $\tilde{W}$ *in* $Q(fm, SL^1)$.

First we show the following two lemmas:

LEMMA 3. *Let* f_i, $i = 1, 2$, *be bounded integrable functions in* $\mathbf{R}^2$ *such that* $f_i \geq 1$ *a.e. in open sets* W_i *and* $f_i = 0$ *a.e. in* W_i^c, *and let* $\Omega_i \in Q(f_i m, SL^1)$, $i = 1, 2$. *If* $f_1 \leq f_2$ *a.e. in* $\mathbf{R}^2$, *then* $\chi_{\Omega_1} \leq \chi_{\Omega_2}$ *a.e. in* $\mathbf{R}^2$.

PROOF. Assume that $\Omega_1 \setminus \Omega_2 \neq \varnothing$. Take a point $y \in \Omega_1 \setminus \Omega_2$ and set

$$s(x) = \begin{cases} g_{\Omega_1}(x, y) & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2 \setminus \Omega_1, \end{cases}$$

where $g_{\Omega_1}(x, y)$ denotes the Green function in Ω_1 with pole at y. Then $s \geq 0$ in $\Omega_2 \cup \Omega_1$, $-s \mid \Omega_1 \in SL^1(\Omega_1)$ and $s \mid \Omega_2 = s^*$ a.e. in Ω_2 for some $s^* \in SL^1(\Omega_2)$, because $m(\Omega_1) \leq m(\Omega_2) < \infty$. Hence

$$\int_{\Omega_1} s\,dm \leq \int sf_1\,dm \leq \int sf_2\,dm \leq \int_{\Omega_2} s\,dm$$

and so

$$\int_{\Omega_1 \setminus \Omega_2} s\,dm \leq \int_{\Omega_2 \setminus \Omega_1} s\,dm = 0.$$

This implies that $m(\Omega_1 \setminus \Omega_2) = 0$, namely, $\chi_{\Omega_1} \leq \chi_{\Omega_2}$ a.e. in $\mathbf{R}^2$.

COROLLARY. *Let* f *be a bounded integrable function in* $\mathbf{R}^2$ *such that* $f \geq 1$ *a.e. in an open set* W *and* $f = 0$ *a.e. in* W^c. *Let* $\Omega_i \in Q(fm, SL^1)$, $i = 1, 2$. *Then* $\chi_{\Omega_1} = \chi_{\Omega_2}$ *a.e. in* $\mathbf{R}^2$.

LEMMA 4. *Let* g *be a bounded nonnegative integrable function in* $\mathbf{R}^2$ *with compact support which is contained in a connected open set* W. *Let* $\int g\,dm > 0$ *and* K *be a compact subset of* W. *Then there are a bounded nonnegative integrable function* $f_{g,K}$ *in* $\mathbf{R}^2$ *and a bounded connected open set* $W_{g,K}$ *such that* $f_{g,K} > 0$ *in* $W_{g,K}$, $f_{g,K} = 0$ *in* $W_{g,K}^c$, $K \cup \operatorname{supp} g \subset W_{g,K} \subset \overline{W_{g,K}} \subset W$ *and* $\int sg\,dm \leq \int sf_{g,K}\,dm$ *for every* $s \in SL^1(W)$.

PROOF. We may assume that $\inf_{x\in L} g(x) > 0$ for a compact subset L of W with $m(L) > 0$. Let δ be a number such that $0 < \delta < d(L, \partial W)/2$, where $d(L, \partial W)$ denotes the distance between L and ∂W, and define a bounded nonnegative integrable function g_1 in $\mathbf{R}^2$ by

$$g_1(x) = \int_{B(x;\,\delta)} g(y)\chi_L(y)\,dm(y)/m(B(x;\delta)).$$

Then g_1 is continuous, $\operatorname{supp} g_1$ is compact and $\int sg\,dm \leqslant \int s(g\chi_{L^c} + g_1)\,dm$ for every $s \in SL^1(W)$. Take a ball B_1 and a number $\alpha_1 > 0$ so that $\overline{B_1} \subset W$ and $g_1 \geqslant \alpha_1$ in B_1. For every $x \in (K \cup \operatorname{supp} g \cup \operatorname{supp} g_1)$, we can find balls B_j, $j = 2, 3, \ldots, n$, with centers p_j such that $p_n = x$, $\overline{B_j} \subset W$ and $p_j \in B_{j-1}$ for every j. Let $\nu_1 = \alpha_1\chi_{B_1}$. Assume that there are a bounded nonnegative integrable function ν_{j-1} in $\mathbf{R}^2$ and a number $\alpha_{j-1} > 0$ such that $\operatorname{supp} \nu_{j-1} \subset \bigcup_{i=1}^{j-1} \overline{B_i}$, $\nu_{j-1} \geqslant \alpha_{j-1}$ in $\bigcup_{i=1}^{j-1} B_i$ and $\int s\nu_{j-1}\,dm \geqslant \int s\nu_1\,dm$ for every $s \in SL^1(W)$. Take a ball B with center p_j such that $B \subset B_{j-1} \cap B_j$. Then

$$\begin{aligned}\int s\nu_{j-1}\,dm &= \int s(\nu_{j-1} - \alpha_{j-1}\chi_B)\,dm + \alpha_{j-1}\int_B s\,dm \\ &\leqslant \int s(\nu_{j-1} - \alpha_{j-1}\chi_B)\,dm + \alpha_{j-1}\frac{m(B)}{m(B_j)}\int_{B_j} s\,dm\end{aligned}$$

for every $s \in SL^1(W)$. Set $\nu_j = \nu_{j-1} - \alpha_{j-1}\chi_B + (\alpha_{j-1}m(B)/m(B_j))\chi_{B_j}$ and $\alpha_j = \alpha_{j-1}m(B)/m(B_j)$. The function ν_j and a number α_j satisfy the above conditions for j. Thus, by induction, we can construct ν_n and $\alpha_n > 0$ such that $\operatorname{supp} \nu_n \subset \bigcup_{j=1}^{n} \overline{B_j}$, $\nu_n \geqslant \alpha_n$ in $\bigcup_{j=1}^{n} B_j$ and $\int s\nu_n\,dm \geqslant \int s\nu_1\,dm$ for every $s \in SL^1(W)$.

Let us write ν_x and V_x for ν_n and $\bigcup_{j=1}^{n} B_j$, respectively. Since $K \cup \operatorname{supp} g \cup \operatorname{supp} g_1$ is compact, we can find a finite number of open sets $V_{x_1}, \ldots, V_{x_k}$ such that $(K \cup \operatorname{supp} g \cup \operatorname{supp} g_1) \subset \bigcup_{j=1}^{k} V_{x_j}$. Set

$$f_{g,K} = g\chi_{L^c} + g_1 - \alpha_1\chi_{B_1} + \frac{1}{k}\sum_{j=1}^{k} \nu_{x_j}, \qquad W_{g,K} = \bigcup_{j=1}^{k} V_{x_j}.$$

These satisfy the required condition.

PROOF OF THEOREM 2. At first, let us construct an open set $G \in Q(fm, SL^1)$. For every connected component O_i of W, let L_i be a compact subset of O_i such that $\int (f-1)\chi_{L_i}\,dm > 0$. Let $g_i = (f-1)\chi_{L_i}$ and let $\{O_{i,j}\}$ be an exhaustion of O_i such that $\overline{O_{i,j}}$ is compact for every j. By using Lemma 4, we can find $f_{i,j} = f_{g_i/2^j, \overline{O_{i,j}}}$ and $W_{i,j} = W_{g_i/2^j, \overline{O_{i,j}}}$ such that $f_{i,j} > 0$ in $W_{i,j}$, $f_{i,j} = 0$ in $W_{i,j}^c$, $\overline{O_{i,j}} \cup L_i \subset W_{i,j} \subset \overline{W_{i,j}} \subset W$ and $\int sg_i/2^j\,dm \leqslant \int sf_{i,j}\,dm$ for every $s \in SL^1(W)$. Set

$$f_0 = f - \sum_{i=1}^{\infty} g_i, \qquad f_n = f_0\chi_{W_n} + \sum_{1\leqslant i\leqslant n}\ \sum_{1\leqslant j\leqslant n-i+1} f_{i,j}, \qquad n = 1, 2, \ldots,$$

where $W_n = \bigcup_{1\leqslant i\leqslant n} \bigcup_{1\leqslant j\leqslant n-i+1} W_{i,j}$. Then f_n is a bounded integrable function in $\mathbf{R}^2$ with $f_n > 1$ in a bounded open set W_n and $f_n = 0$ in W_n^c.

From the argument given in §1, we can construct the minimum open set $\tilde{W}_n \in Q(f_n m, SL^1)$ for every n. Since $f_n \leqslant f_{n+1}$, from the proof of Lemma 2, we obtain $u(f_n) \leqslant u(f_{n+1})$ (for the notation, see the proof of Lemma 2). Hence $\tilde{W}_n \subset \tilde{W}_{n+1}$. Set $G = \bigcup \tilde{W}_n$. By the proof of Proposition 3, we have $m(W_n \setminus \tilde{W}_n) = 0$. Hence it follows that $m(W \setminus G) = 0$.

Next let us show

$$\int sf\,dm \leqslant \int_G s\,dm$$

for every $s \in SL^1(G)$. For every $\varepsilon > 0$, we can take a number n so that

$$\int_G s\,dm + \varepsilon \geqslant \int_{\tilde{W}_n} s\,dm$$

and

$$\int sf\,dm - \varepsilon \leqslant \int s\Big(f_0\chi_{W_n} + \sum_{1\leqslant i\leqslant n}\ \sum_{1\leqslant j\leqslant n-i+1} g_i/2^j \Big)\,dm.$$

Since

$$\int s\Big(f_0\chi_{W_n} + \sum_{1\leqslant i\leqslant n}\ \sum_{1\leqslant j\leqslant n-i+1} g_i/2^j \Big)\,dm \leqslant \int s\Big(f_0\chi_{W_n} + \sum_{1\leqslant i\leqslant n}\ \sum_{1\leqslant j\leqslant n-i+1} f_{i,j} \Big)\,dm$$
$$\leqslant \int_{W_n} sf_n\,dm \leqslant \int_{\tilde{W}_n} s\,dm,$$

we have

$$\int sf\,dm \leqslant \int_G s\,dm + 2\varepsilon$$

for every $\varepsilon > 0$. Hence

$$\int sf\,dm \leqslant \int_G s\,dm$$

for every $s \in SL^1(G)$.

For $s = 1$, we have

$$\int f\,dm = \lim_{n\to\infty} \int \Big(f_0\chi_{W_n} + \sum_{1\leqslant i\leqslant n}\ \sum_{1\leqslant j\leqslant n-i+1} g_i/2^j \Big)\,dm$$
$$= \lim_{n\to\infty} \int f_n\,dm = \lim_{n\to\infty} m\big(\tilde{W}_n\big) = m(G).$$

Hence $m(G) < \infty$. Thus we have proved that $G \in Q(fm, SL^1)$.

From the corollary to Lemma 3, $\chi_\Omega = \chi_G$ a.e. for every $\Omega \in Q(fm, SL^1)$. Since $\chi_\Omega - f$ has not necessarily compact support, take two distinct points ζ_1 and ζ_2 in $(\bigcup \Omega)^c$, where $\bigcup \Omega$ denotes the union of all $\Omega \in Q(fm, SL^1)$, consider the generalized logarithmic potential $U^{\chi_\Omega - f}(x; \zeta_1, \zeta_2)$ (see [**8**, §3]) and set

$$u(x) = -\frac{1}{2\pi} U^{\chi_\Omega - f}(x; \zeta_1, \zeta_2).$$

The function u is determined independently of the choice of $\Omega \in Q(fm, SL^1)$. Let $\tilde{W} = \{x \in \mathbf{R}^2 \mid u(x) > 0\}$. If $\Omega \in Q(fm, SL^1)$ and $x \notin \Omega$, then $u(x) = 0$ and so

$x \notin \tilde{W}$. Therefore $\tilde{W} \subset \Omega$ for every $\Omega \in Q(fm, SL^1)$. Since $u(x) = 0$ in $\tilde{W}^c$, $\Delta u = 0$ a.e. on $\tilde{W}^c$. Hence $0 = \Delta u = \chi_\Omega - f$ a.e. in $\tilde{W}^c$ and so $\chi_\Omega \leq \chi_{\tilde{W}}$ a.e. in $\mathbf{R}^2$. This implies that $\tilde{W} \subset \Omega$ and $m(\Omega \setminus \tilde{W}) = 0$ for every $\Omega \in Q(fm, SL^1)$.

Finally, the fact that $\tilde{W} \in Q(fm, SL^1)$ follows from the similar argument given in the proof of Proposition 4. In contrast with the proof of Proposition 4, the open set $\tilde{W}$ is not necessarily bounded. For the generalized logarithmic potential and the similar argument given in the proof of Proposition 4, see **[8**, §3].

3. The case of higher dimensions. Our theorem is also valid for the case of higher dimensions. In the case of dimension $d \geq 3$, let us write by S_d the surface area of the $(d-1)$-dimensional unit hypersphere, namely, $S_d = 2\pi^{d/2}/\Gamma(d/2)$. We replace $-\log|x-y|$ by $|x-y|^{2-d}$ and consider the Newton potential

$$U^g(y) = \int |x-y|^{2-d} g(x)\, dm(x)$$

instead of the logarithmic potential which we have used in the case of dimension 2. In the above definition, g is a real-valued bounded integrable function defined in $\mathbf{R}^d$ and m denotes the d-dimensional Lebesgue measure.

It is known that

(1) U^g is of class C^1,

(2) $\partial U^g(x)/\partial x_j - \partial U^g(y)/\partial x_j = O(|x-y|\log(1/|x-y|))$, $j = 1, 2, \ldots, d$, for every pair of points x and y with $|x-y| < e^{-2}$.

(3) $\Delta U^g = -(d-2)S_d g$ in the sense of distributions.

Therefore our arguments are also valid if we replace $-1/(2\pi)$ and $-\log|x-y|$ by $-1/((d-2)S_d)$ and $|x-y|^{2-d}$, respectively.

Let us give here a remark on the generalized logarithmic potential used in the proof of Theorem 2. It is unnecessary to consider "generalized" in the case of dimension $d \geq 3$. Because we can define the Newton potential U^g of a bounded integrable function g which has not necessarily a compact support.

4. Hele-Shaw flows with a free boundary. As an application of the new proof of our theorem, we deal with Hele-Shaw flows with a free boundary produced by the injection of fluid into the narrow gap between two parallel planes (for the mathematical formulation, see Richardson **[6]** and Sakai [7]).

In [7], the author has defined a weak solution of a free boundary problem of Hele-Shaw flows with the initial connected open set $\Omega(0)$. It is a family $\{\Omega(t)\}_{t \geq 0}$ of quadrature domains $\Omega(t)$ such that $\Omega(t)$ is the minimum domain in $Q(\chi_{\Omega(0)}m + t\delta_c, SL^1)$ for every $t > 0$, where δ_c denotes the Dirac measure at the injection point $c \in \Omega(0)$ of the fluid.

Recently, Gustafsson **[3]** has defined another weak solution of Hele-Shaw flows by using variational inequalities (for the case having the container wall, see Elliott and Janovský **[2]**).

Let $f_t = \chi_{\Omega(0)} + t(1/m(B(c; r)))\chi_{B(c;r)}$ (in [3], Gustafsson has used $2\pi t$ and $B(0; r)$ for t and $B(c; r)$, respectively), where $\Omega(0)$ denotes a bounded connected open set and $B(c; r)$ satisfies $\overline{B(c; r)} \subset \Omega(0)$, and consider the variational problem given before Proposition 2 for large ball B_t (which depends on t) and for a function

$\psi_t = \psi(f_t)$. Then Gustafsson's weak solution $\{\Omega(t)\}_{t\geq 0}$ is, in our notation given in the proof of Lemma 2, a family of domains $\Omega(t) = \Omega(0) \cup \Omega(f_t)$ for every $t > 0$.

In this section, we shall note first that $\Omega(t) = \Omega(f_t)$, namely, $\Omega(0) \subset \Omega(f_t)$ (this result is also given by Gustafsson [3, Lemma 14(iv)]) and next show that the above two weak solutions are identical with each other.

The first assertion follows immediately from the following lemma:

LEMMA 5. *Let f, W and $\tilde{W}$ be as in Theorem 2. Then $W \subset \tilde{W}$.*

PROOF. Since $f \geq 1$ a.e. in W, $\Delta u = \chi_{\tilde{W}} - f \leq 0$ a.e. in W. Hence u is a nonnegative superharmonic function in W. If $u(x) = 0$ for some $x \in W$, then $u \equiv 0$ in the connected component of W containing x. This contradicts $m(W \setminus \tilde{W}) = 0$ and so $u(x) > 0$ in W, namely, $W \subset \tilde{W}$.

The next corollary guarantees that $\Omega(f_t)$ is connected.

COROLLARY. *If W is connected, then $\tilde{W}$ is also connected.*

PROOF. Assume that $\tilde{W}$ is disconnected. Since $W \subset \tilde{W}$ and W is connected, we can find a connected component O of $\tilde{W}$ such that $W \cap O = \varnothing$. For every $s \in SL^1(\tilde{W} \setminus O)$, let $\tilde{s}$ be a function defined by $\tilde{s}(x) = s(x)$ in $\tilde{W} \setminus O$ and $\tilde{s}(x) = 0$ in O. Then $\tilde{s} \in SL^1(\tilde{W})$ and

$$\int_W sf\,dm = \int_W \tilde{s}f\,dm \leq \int_{\tilde{W}} \tilde{s}\,dm = \int_{\tilde{W}\setminus O} s\,dm.$$

Hence $\tilde{W} \setminus O \in Q(fm, SL^1)$. This contradicts the fact that $\tilde{W}$ is the minimum domain in $Q(fm, SL^1)$.

To show the second assertion, by the argument given in §1, it is sufficient to show that $Q(\chi_{\Omega(0)}m + t\delta_c, SL^1) = Q(f_t m, SL^1)$ for every $t > 0$. This follows immediately from the proposition below.

For the sake of simplicity, we assume that W is a connected open set. Let μ be a positive finite measure with compact support contained in W. For a number α with $0 < \alpha < d(\operatorname{supp}\mu, \partial W)/2$, where $d(\operatorname{supp}\mu, \partial W)$ denotes the distance between $\operatorname{supp}\mu$ and ∂W, let us define a bounded function $M_\alpha\mu$ by

$$(M_\alpha\mu)(x) = \frac{\mu(B(x;\alpha))}{m(B(x;\alpha))}.$$

The support of $M_\alpha\mu$ is contained in W.

LEMMA 6. $Q(\chi_W m + \mu, SL^1) = Q((\chi_W + M_\alpha\mu)m, SL^1)$.

PROOF. We may assume that $\mu \neq 0$. If $\Omega \in Q((\chi_W + M_\alpha\mu)m, SL^1)$, then, by Lemma 5, $W \subset \Omega$. Since

$$\int s\,d\mu \leq \int s(M_\alpha\mu)\,dm$$

for every $s \in SL^1(W)$, $\Omega \in Q(\chi_W m + \mu, SL^1)$.

Conversely, assume that $\Omega \in Q(\chi_W m + \mu, SL^1)$. Set $G = \{x \in W \mid (M_\alpha \mu)(x) > 0\}$. Then, since $M_\alpha \mu$ is lower semicontinuous, G is an open set containing $\operatorname{supp} \mu$. We shall show $\overline{G} \subset \Omega$. If $y \in \overline{G} \setminus \Omega$, then $\mu(B(y; \beta)) > 0$ for β with $\alpha < \beta < d(\operatorname{supp} \mu, \partial W)/2$. Set

$$s(x) = \max\{\log(1/|x-y|), \log(1/\beta)\} - \log(1/\beta).$$

Then $s \mid \Omega \in SL^1(\Omega)$. Since $m(W \setminus \Omega) = 0$,

$$\int s(\chi_W dm + d\mu) > \int_W s\, dm = \int_\Omega s\, dm.$$

This contradicts $\Omega \in Q(\chi_W m + \mu, SL^1)$. Hence $\overline{G} \subset \Omega$.

Let $s \in SL^1(\Omega)$, and let $s^* \in SL^1(\Omega)$ be harmonic in G and satisfy $s^* = s$ a.e. in $\Omega \setminus G$. Since

$$\int_W s^*(\chi_W + M_\alpha \mu)\, dm = \int_W s^*(\chi_W dm + d\mu) \leqslant \int_\Omega s^*\, dm$$

and $s \leqslant s^*$ in G, we have

$$\int_W s(\chi_W + M_\alpha \mu)\, dm \leqslant \int_W s^*(\chi_W + M_\alpha \mu)\, dm + \int_G (s - s^*)\, dm$$
$$\leqslant \int_\Omega s^*\, dm + \int_G (s - s^*)\, dm = \int_\Omega s\, dm.$$

Therefore $\Omega \in Q((\chi_W + M_\alpha \mu)m, SL^1)$.

PROPOSITION 5. *Let μ_i, $i = 1, 2$, be positive finite measures with compact support contained in a connected open set W. If there is an open subset G of W such that $G \supset \operatorname{supp} \mu_1 \cup \operatorname{supp} \mu_2$ and $\int h\, d\mu_1 = \int h\, d\mu_2$ for every harmonic function in G, then $Q(\chi_W m + \mu_1, SL^1) = Q(\chi_W m + \mu_2, SL^1)$.*

PROOF. By Lemma 6, it is sufficient to show that $Q((\chi_W + M_\alpha \mu_1)m, SL^1) = Q((\chi_W + M_\alpha \mu_2)m, SL^1)$ for small $\alpha > 0$. We obtain this equality by using Lemma 5 and the argument as in the proof of Lemma 6.

5. Quadrature domains for harmonic and analytic functions. In [7], quadrature domains for harmonic and analytic functions are introduced. Let ν be a positive finite measure in $\mathbf{R}^2$ and let $HL^1(\Omega)$ (resp. $AL^1(\Omega)$) be the class of all real-valued (resp. complex-valued) harmonic (resp. analytic) integrable functions in Ω. A nonempty open set Ω is called a quadrature domain of class HL^1 (resp. AL^1), if Ω satisfies (Qi), (Qiii) and

(Qii)′ $$\int_\Omega |h|\, d\nu < \infty \quad \text{and} \quad \int_\Omega h\, d\nu = \int_\Omega h\, dm$$

for every $h \in HL^1(\Omega)$ (resp. $h \in AL^1(\Omega)$). We denote by $Q(\nu, HL^1)$ (resp. $Q(\nu, AL^1)$) the class of all quadrature domains of ν for class HL^1 (resp. AL^1).

By using the generalized logarithmic potential, we obtain the following proposition:

PROPOSITION 6. *Let f and W be as in Theorem 2. Let Ω be an open set with finite area, let ζ_1 and ζ_2 be two distinct points in Ω^c and set $u(x) = -1/(2\pi)U^{\chi_\Omega - f}(x; \zeta_1, \zeta_2)$. Then*

(1) $\Omega \in Q(fm, SL^1)$ *if and only if* $u = 0$ *in* Ω^c *and* $u \geq 0$ *in* Ω,

(2) $\Omega \in Q(fm, HL^1)$ *if and only if* $u = 0$ *and* $\partial u/\partial x_j = 0, j = 1, 2,$ *in* Ω^c,

(3) $\Omega \in Q(fm, AL^1)$ *if and only if* $\partial u/\partial x_j = 0, j = 1, 2,$ *in* Ω^c.

PROOF. The assertions (1) and (2) are proved from the argument similar to the proof of Proposition 4. Let $(\chi_\Omega - f)\hat{}$ be the generalized Cauchy transform of $\chi_\Omega - f$ (for the definition, see [8]). Then $(\chi_\Omega - f)\hat{} = (\partial/\partial x_1 - i\partial/\partial x_2)U^{\chi_\Omega - f}$. Hence $\partial u/\partial x_j = 0, j = 1, 2$, in Ω^c implies that $(\chi_\Omega - f)\hat{} = 0$ in Ω^c. Let $z = x_1 + ix_2$. Since the subclass of $AL^1(\Omega)$ which consists of all linear combinations of $1/(z - \zeta_k)$ with $\zeta_k \in \Omega^c$ is dense in $AL^1(\Omega)$ (see Bers [1]), the assertion (3) follows.

REFERENCES

1. L. Bers, *An approximation theorem*, J. Analyse Math. **14** (1965), 1–4.

2. C. M. Elliott and V. Janovský, *A variational inequality approach to Hele-Shaw flow with a moving boundary*, Proc. Roy. Soc. Edinburgh Sect. A **88** (1981), 93–107.

3. B. Gustafsson, *Applications of variational inequalities to a moving boundary problem for Hele Shaw flows*, TRITA-MAT-1981-9, Mathematics, Roy. Inst. Tech., Stockholm, p. 84.

4. L. I. Hedberg, *Approximation in the mean by solutions of elliptic equations*, Duke Math. J. **40** (1973), 9–16.

5. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

6. S. Richardson, *Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel*, J. Fluid Mech. **56** (1972), 609–618.

7. M. Sakai, *Quadrature domains*, Lecture Notes in Math., vol. 934, Springer-Verlag, Berlin, 1982.

8. ______, *Null quadrature domains*, J. Analyse Math. **40** (1981), 144–154.

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